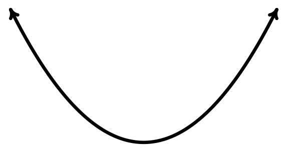
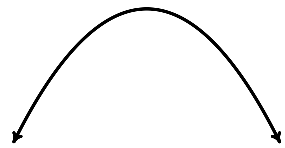
MATH 1650: POLYNOMIAL FUNCTIONS SUMMARY SHEET

MONOMIAL FUNCTIONS: Monomial functions consist of the constant functions along with functions of the form $f(x) = ax^n$ for natural numbers n. (i.e., $n = 1, 2, 3, \dots$)

EVEN DEGREE MONOMIAL FUNCTIONS: For $n \geq 2$, the graph of $f(x) = ax^n$ resembles:

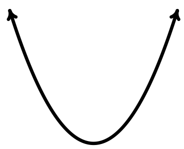


$a > 0$

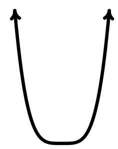


$a < 0$

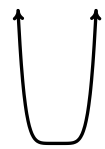
As n increases, the curve gets 'steeper' at the sides and 'flatter' at the vertex:



$y = x^2$



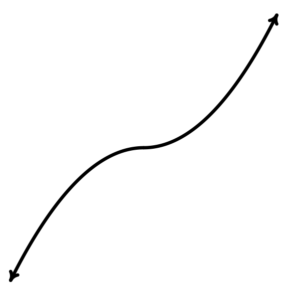
$y = x^4$



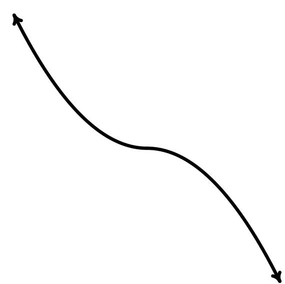
$y = x^6$

EVEN FUNCTIONS: A function is called **even** if $f(-x) = f(x)$ for all x in the domain of f.
NOTE: The graphs of even functions $y = f(x)$ are symmetric about the y-axis.

ODD DEGREE MONOMIAL FUNCTIONS: For $n \geq 3$, the graph of $f(x) = ax^n$ resembles:

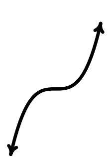


$a > 0$

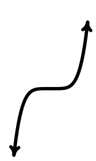


$a < 0$

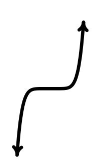
As n increases, the curve gets 'steeper' at the sides and 'flatter' in the middle:



$y = x^3$



$y = x^5$



$y = x^7$

ODD FUNCTIONS: A function is called **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
NOTE: The graphs of odd functions $y = f(x)$ are symmetric about the origin.

POLYNOMIAL FUNCTION: A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, . . . , a_n are real numbers and n is a whole number.

NOTE: For $n > 0$, the subscripts here match the power of x. For example, a_1 is the coefficient of $x = x^1$, a_2 is the coefficient of x^2, etc. Constant, linear, and quadratic functions are all part of the larger 'polynomial' family.

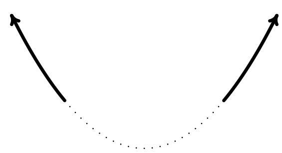
TERMINOLOGY: A polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$, with $a_n \neq 0$ has:

- **degree** n
- **leading term** $a_n x^n$
- **leading coefficient** a_n

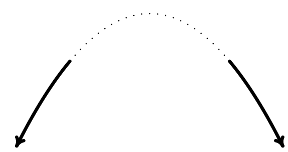
END BEHAVIOR: The **end behavior** of a function $f(x)$ describes the behavior of the $f(x)$ as $x \to \pm\infty$.

THEOREM: The end behavior of polynomial functions is determined by its leading term:

EVEN DEGREE: n is even: $n = 2, 4, 6, \ldots$

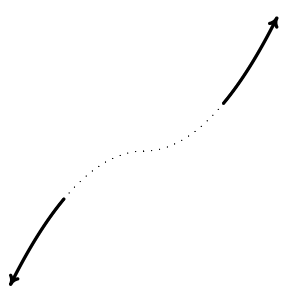


$a_n > 0$
as $x \to \pm\infty$, $f(x) \to \infty$



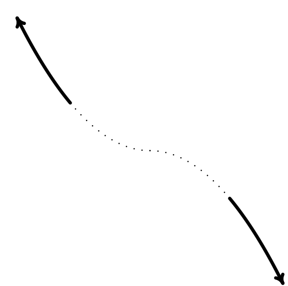
$a_n < 0$
as $x \to \pm\infty$, $f(x) \to -\infty$

ODD DEGREE: n is odd: $n = 1$ (linear), $3, 5, \ldots$



$a_n > 0$

as $x \to -\infty$, $f(x) \to -\infty$,
as $x \to \infty$, $f(x) \to \infty$



$a_n < 0$

as $x \to -\infty$, $f(x) \to \infty$,
as $x \to \infty$, $f(x) \to -\infty$

ZEROS OF POLYNOMIALS:

DEFINITION: The **zeros** of a function f are the solutions to the equation $f(x) = 0$.

NOTE: Geometrically, c is a zero of f means $(c, 0)$ is an x-intercept of the graph of $y = f(x)$.

MULTIPLICITY OF ZEROS: Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x = c$ is a zero of **multiplicity** m.

NOTE: In other words, when solving $f(x) = 0$, a zero of multiplicity m is a solution which is repeated 'm' times.

EXAMPLES:

- Solving $f(x) = x^3(x-3)^2(x+2) = 0$, we get $x = 0$ three times, $x = 3$ twice, and $x = -2$ once.

  Therefore, $x = 0$ has multiplicity 3, $x = 3$ has multiplicity 2, and $x = -2$ has multiplicity 1.

  NOTE: We can rewrite: $f(x) = x^3(x-3)^2(x+2) = (x-0)^3(x-3)^2(x-(-2))^1$.

- Solving $G(t) = (2t-1)^3(t+1)^3(4-4t^2) = 0$, we get $x = \frac{1}{2}$ three times, $t = -1$ four times total: three times from $(t+1)^3 = 0$ and once from $(4-4t^2) = 0$, and, finally, $t = 1$ once.

  Therefore, $t = \frac{1}{2}$ has multiplicity 3, $t = -1$ has multiplicity 4, and $t = 1$ has multiplicity 1.

  NOTE: We can rewrite: $G(t) = (2t-1)^3(t+1)^3(4-4t^2) = -32\left(t - \dfrac{1}{2}\right)^3 (t-(-1))^4(t-1)^1$.

THE ROLE OF MULTIPLICITY: Suppose f is a polynomial function and c is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

NOTE: The graph near a zero of multiplicity m resembles the power function $y = ax^m$.

As a result, the higher the multiplicity, the flatter the graph near the intercept.
DEFINITION: Suppose f is a function with $f(a) = b$.

- We say f has a **local maximum** at the point (a, b) if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f.'
- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f.'

NOTE: Said differently, f has a local maximum (minimum) at (a, b) if there are no points higher (lower) on the graph than (a, b) 'near' $x = a$. An extreme example of this is a constant function $f(x) = b$. Every point on the graph $y = b$ is both a local maximum and a local minimum! The local maximums and minimums of the graph of a function are called collectively the **local extrema** of the graph.

MATH 1650 ZEROS OF POLYNOMIALS POWER TOOLS

Below is a list of all the 'mathematical power tools' that help us find all the zeros of a polynomial function, p.

- **STRATEGIES FROM INTERMEDIATE ALGEBRA:** Don't forget special forms:
  - Does the polynomial factor by grouping?
  - Is the polynomial a 'quadratic in disguise'? That is:
    - The polynomial has exactly three terms.
    - The exponent on one term is exactly twice the exponent on the other.
- **THE REMAINDER THEOREM** says if you divide $p(x)$ by $(x - c)$ and get remainder r, then $p(c) = r$. This means the point (c, r) is on the graph of $y = p(x)$. In particular, if $r = 0$, then we've found an x-intercept, $(c, 0)$. This leads us to ...
- **THE FACTOR THEOREM** is a special case of The Remainder Theorem. It tells you the real number c is a zero if and only if when you divide $p(x)$ by $(x - c)$, the remainder is 0.

  NOTE: This lets you partially factor the polynomial into a linear term $(x - c)$ times a polynomial with degree one less than what you started. Continue this process until you're left with a quadratic.
- **CAUCHY'S BOUND** gives you the Xmin and Xmax of your calculator window. Take the 'biggest' (in absolute value) of the non-leading coefficients and divide by the leading coefficient. Call this number M. Then all the real zeros are between $-|M| - 1$ and $|M| + 1$.
- **THE RATIONAL ZEROS THEOREM** gives you a list of 'friendly' numbers to try. They are of the form:

$$\pm\frac{\text{factors of the constant term}}{\text{factors of the leading coefficient}}$$

- **THE FUNDAMENTAL THEOREM OF ALGEBRA** says that every polynomial has at least one complex zero. This zero may be real or non-real, and it may not even be expressible using 'common' notation.
- **THE n ZEROS THEOREM** states that, counting multiplicities, a polynomial of degree n has n complex zeros. Again, some may be real, some non-real, and they may not be expressible using 'common' notation.
- **THE COMPLEX CONJUGATE PAIRS THEOREM** states that, for polynomial functions with real number coefficients, non-real zeros come in complex conjugate pairs: $a + bi$ and $a - bi$.
- **THE REAL FACTORIZATION THEOREM** states that, in theory, every polynomial function with real number coefficients can be factored into a product of (repeated) linear factors which correspond to the real zeros of p and irreducible quadratic factors which correspond to the non-real zeros of p.
- **THE COMPLEX FACTORIZATION THEOREM** states that, in theory, every polynomial function can be completely completely factored over the complex numbers as follows:

$$p(x) = (\text{leading coefficient})(x - \text{each zero})^{\text{corresponding multiplicity}}$$

- If all else fails, a graphing utility to get an *approximation* of the real zeros.